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On the Focal Surfaces of the Congruences of Tangents to a given Surface.

By A. Pell.

§1.

Consider a surface (Σ) referred to its lines of curvature u and v. For brevity we shall call the u-line and the v-line those lines for which $v =$ const. and $u =$ const. respectively. The axis of x is tangent to the u-line and drawn in the direction of the increasing arc; the axis of y is tangent to the v-line, and its direction is such that a rotation around the axis of z, bringing the x-axis into coincidence with the y-axis, is represented by a line directed along the positive part of the z-axis. The direction of the z-axis is that of the normal to the surface.

Using Darboux's notation, we have for (Σ) the following formulæ:*

$$\left.\begin{array}{l} \xi = \sqrt{E},\ \eta = 0,\ \xi_1 = 0,\ \eta_1 = \sqrt{G},\ p = 0, \\ q_1 = 0, \quad r = -\dfrac{1}{\sqrt{G}} \cdot \dfrac{\partial \sqrt{E}}{\partial v},\ r_1 = \dfrac{1}{\sqrt{E}} \cdot \dfrac{\partial \sqrt{G}}{\partial u},\ \zeta = 0,\ \zeta_1 = 0. \\ \dfrac{\partial r}{\partial v} - \dfrac{\partial r_1}{\partial u} = -q p_1,\ \dfrac{\partial p_1}{\partial u} = -q r_1,\ \dfrac{\partial q}{\partial v} = r p_1, \\ \dfrac{\partial}{\partial u}\left(\dfrac{1}{q} \cdot \dfrac{\partial p_1}{\partial u}\right) + \dfrac{\partial}{\partial v}\left(\dfrac{1}{p_1} \cdot \dfrac{\partial q}{\partial v}\right) + q p_1 = 0. \end{array}\right\} \quad (1)$$

Here ξ, η, ζ and ξ_1, η_1, ζ_1 are the components of the velocity of translation of the origin of the movable axes relatively to these axes when u or v vary respectively, and p, q, r, p_1, q_1, r_1 are the rotations around the axes, the first three when only u varies, the last three when only v varies.

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* Darboux, "Théorie générale des surfaces," vol. II, p. 386.

Then calling ρ_1 and ρ_2 the principal radii of curvature of the u-line and the v-line respectively, we have

$$\rho_1 = -\frac{\sqrt{E}}{q_1}, \quad \rho_2 = \frac{\sqrt{G}}{p_1}. \tag{2}$$

Further, let h_1 and h_2 be the principal radii of curvature of the u- and the v-lines; τ_1 and τ_2 their radii of torsion; R_2 and R_1 the radii of geodesic curvatures, directed along the positive parts of the y and the x axes respectively. For these radii we have

$$r = \frac{\sqrt{E}}{R_2}, \quad r_1 = -\frac{\sqrt{G}}{R_1}, \tag{3}$$

and since

$$\frac{\sqrt{E}}{t_1} = -p = 0; \quad \frac{\sqrt{G}}{t_2} = -q_1 = 0, \tag{4}$$

where t_1 and t_2 are the radii of geodesic torsions of the u- and the v-lines respectively, we have

$$\frac{1}{\tau_1} = \frac{1}{\sqrt{E}} \cdot \frac{\partial \varpi_1}{\partial u}, \quad \frac{1}{\tau_2} = \frac{1}{\sqrt{G}} \cdot \frac{\partial \varpi_2}{\partial v}. \tag{5}$$

Here ϖ_1 and ϖ_2 are the angles between the radii of principal curvatures of the surface (Σ) and the segments h_1 and h_2.

We have

$$\left.\begin{aligned} h_1 &= \rho_1 \cos \varpi_1, & h_2 &= \rho_2 \cos \varpi_2, \\ h_1 &= R_2 \sin \varpi_1, & h_2 &= R_1 \sin \varpi_2, \\ \frac{1}{h_1^2} &= \frac{1}{R_2^2} + \frac{1}{\rho_1^2}, & \frac{1}{h_2^2} &= \frac{1}{R_1^2} + \frac{1}{\rho_2^2}. \end{aligned}\right\} \tag{6}$$

§2.

Our consideration of the focal surfaces of the congruences of the tangents to the lines of curvature of the surface (Σ), is based on the following two theorems given by Darboux* and Koenigs.†

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* L. c. vol. III, p. 121.

† G. Koenigs, "Sur les propriétés infinitésimal de l'espace réglé," Ann. de l'école normale, etc. 1882, p. 248.

Theorem I.—The locus of the centers of geodesic curvature of lines of curvature of any surface is the edge of regression of the developable surface, generated by the tangent planes of the surface at all points of the lines of curvature.

This theorem shows that the distances between the point M of (Σ) and the corresponding points M' of the focal surfaces under discussion are: for the family $v =$ const., R_1, and for the family $u =$ const., R_2.

Theorem II.—The edges of regression of the developable surfaces of a congruence form two families of curves on the focal surfaces (say S_A and S_B corresponding to the focal surfaces A and B), the osculating planes of which are tangent to the surfaces B and A respectively, and the points of contact describe on these surfaces two families of conjugate lines S_A and S_B.

This theorem states that the osculating plane of u-line is tangent to the focal surface (S_1) and that of the v-line is tangent to (S_2); and further, that the normals to these surfaces are parallel to the binormals of the u- and v-lines respectively.

Hence the study of the focal surface (S_1) of the congruence of the tangents to the u-line can be reduced to the consideration of the motion of a trihedron formed at the center of geodesic curvature of the v-line by the tangent to the u-line, the line parallel to the segment h_1 and the parallel to the binormal, drawn in a suitable direction. The motion of this trihedron we are going to determine from that of the trihedron formed by the tangents to the u- and the v-lines and the normal to the surface (Σ).

The study of the motion of the trihedron formed at the center of geodesic curvature of the u-line by the tangent to the v-line, the parallel to h_2 and the parallel to the binormal of the v-line, will give us means of discussing the focal surface (S_2).

Let us call the trihedron connected with the surface (Σ), T. Consider a point (x, y, z) connected with this trihedron. The projections δx, δy, δz on its axis of an infinitely small displacement of this point are:

$$\left.\begin{aligned}\delta x &= dx + \sqrt{E}\,du + zq\,du - y\,(r du + r_1 dv)\,,\\ \delta y &= dy + \sqrt{G}\,dv + x\,(r du + r_1 dv) - zp_1\,dv\,,\\ \delta z &= dz + yp_1\,dv - xq\,du\,.\end{aligned}\right\} \qquad (7)$$

Suppose now that this point coincide in the first place with the center of geodesic curvature of the v-line ($x = R_1$, $y = 0$, $z = 0$), and in the second place with that of the u-line ($x = 0$, $y = R_2$, $z = 0$). We know that

$$\sqrt{E}\,du = ds_1, \qquad \sqrt{G}\,dv = ds_2,$$

s_1 and s_2 being the lengths of the arcs of the u- and the v-lines respectively. The above displacements take the form

$$\begin{array}{llll} \text{for } (S_1), & & \text{for } (S_2), & \\ \delta x_1 = & dR_1 + ds_1, & \delta x_2 = - & R_2 r_1\,dv \;, \\ \delta y_1 = & R_1\,r\,du \;, & \delta y_2 = & dR_2 + ds_2 , \\ \delta z_1 = - & R_1\,q\,du \;, & \delta z_2 = & R_2\,p_1\,dv \;. \end{array}$$

Substituting for r, q, r_1, p_1, their values, we get

$$\begin{array}{ll} \delta x_1 = dR_1 + ds_1, & \delta x_2 = \dfrac{R_2}{R_1}\,ds_2 \;, \\ \delta y_1 = \dfrac{R_1}{R_2}\,ds_1 \;, & \delta y_2 = d\,R_2 + ds_2, \\ \delta z_1 = \dfrac{R_1}{\rho_1}\,ds_1 \;, & \delta z_2 = \dfrac{R_2}{\rho_2}\,ds_2 \;. \end{array}$$

These displacements are with respect to the trihedron (T). Codazzi's formulæ, as given in Darboux's "Leçons, etc.," require that the tangent plane to the surface contain the axes x and y. But according to theorem II, the osculating plane of the u-line is tangent to (S_1) at the corresponding point; that of the v-line is tangent to (S_2); therefore we must find the values of the projections of the above displacements on the axes of the trihedrons (T_1) and (T_2) formed as stated above. The rotations of (T_1) and (T_2) are obtained by compounding the latter and the rotations through the angles ϖ_1 and ϖ_2 around the axes x and y respectively.

Following is a table giving the direction cosines of the new axes and the old ones.

	x'	y'	z'	x''	y''	z''
x	1	0	0	$\sin \varpi_2$	0	$-\cos \varpi_2$
y	0	$\sin \varpi_1$	$\cos \varpi_1$	0	1	0
z	0	$\cos \varpi_1$	$-\sin \varpi_1$	$\cos \varpi_2$	0	$\sin \varpi_2$

Hence if $\delta x_1'$, $\delta y_1'$, $\delta z_1'$ and $\delta x_2''$, $\delta y_2''$, $\delta z_2''$ are the displacements with respect to (T_1) and (T_2) respectively, we have

$$\begin{aligned} \delta x_1' &= \delta x_1 \cos(xx') + \delta y_1 \cos(yx') + \delta z' \cos(zx) = \delta x_1, \\ \delta y_1' &= \delta y_1 \sin \varpi_1 + \delta z_1 \cos \varpi_1, \\ \delta z_1' &= \delta z_1 \cos \varpi_1 - \delta z_1 \sin \varpi_1, \end{aligned}$$

and after substitution,

$$\left.\begin{aligned} \delta x_1' &= dR_1 + ds_1, \\ \delta y_1' &= \frac{R_1}{h_1} ds_1 \quad , \\ \delta z_1' &= 0 \quad , \end{aligned}\right\} \text{for } (S_1), \tag{8}$$

and

$$\left.\begin{aligned} \delta x_2'' &= \frac{R_2}{h_2} ds_2 \quad , \\ \delta y_2'' &= dR_2 + ds_2, \\ \delta z_2'' &= 0 \quad , \end{aligned}\right\} \text{for } (S_2). \tag{9}$$

These projections we could have obtained at once by remarking that, for example, the focal surface (S_1) can be generated by the motion of the trihedron (T_1) along the u-line, and then considering in the formulæ for the displacements v as a variable parameter. In this case we have

For (T_1),

$$dv = 0; \ \xi\, du = ds_1; \ \eta = 0, \ \zeta = 0,$$
$$p = -\frac{1}{\tau_1}, \ q = 0, \ r = \frac{1}{h_1}.$$

For (T_2),

$$du = 0, \ \eta_1 dv = ds_2, \ \xi_1 = 0, \ \zeta_1 = 0,$$
$$p_1 = 0, \quad q_1 = \frac{1}{\tau_2}, \ r_1 = -\frac{1}{h_2}.$$

The projections of the displacements on the axes of x, y, z now are:

$$\left.\begin{aligned} \delta x_1' &= dx - \frac{y}{h_1} . ds_1 + ds_1, \\ \delta y_1' &= dy + \left(\frac{x}{h_1} + \frac{z}{\tau_1}\right) ds_1, \\ \delta z_1' &= dz - \frac{y}{\tau_1} ds_1 \end{aligned}\right\} \tag{10}$$

and

$$\left.\begin{aligned} \delta x_2'' &= dx + \left(\frac{z}{\tau_2} + \frac{y}{h_2}\right) ds_2, \\ \delta y_2'' &= dy + ds_2 + \frac{x}{h_2}\, ds_2, \\ \delta z_2'' &= dz + \frac{x}{\tau_2}\, ds_2. \end{aligned}\right\} \tag{11}$$

Applying these formulæ to the points $(R_1, 0, 0)$ and $(0, R_2, 0)$, we get

$$\left.\begin{aligned} \delta x_1' &= dR_1 + ds_1, \\ \delta y_1' &= \frac{R_1}{h_1}\,.\, ds_1, \\ \delta z_1' &= 0, \\ \text{and} \qquad \delta x_2'' &= \frac{R_2}{h_2}\, ds_2, \\ \delta y_2'' &= dR_2 + ds_2, \\ \delta z_2'' &= 0. \end{aligned}\right\} \tag{12}$$

§3.

The expressions for the linear elements of the arcs ds_1' and ds_2' of the focal surfaces (S_1) and (S_2) are:

$$\left.\begin{aligned} ds_1'^2 &= \Sigma \delta x'^2 = (dR_1 + ds_1)^2 + \frac{R_1^2}{h_1^2}\, ds_1^2, \\ ds_2'^2 &= \Sigma \delta x''^2 = (dR_2 + ds_2)^2 + \frac{R_2^2}{h_2^2}\, ds_2^2. \end{aligned}\right\} \tag{13}$$

Hence, for (S_1),

$$\begin{aligned} E_1 &= \left(\frac{\partial R_1}{\partial u} + \sqrt{E}\right)^2 + \left(\frac{R_1}{h_1}\right)^2 E, \\ G_1 &= \left(\frac{\partial R_1}{\partial v}\right)^2, \\ F_1 &= \frac{\partial R_1}{\partial v}\left(\frac{\partial R_1}{\partial u} + \sqrt{E}\right), \end{aligned}$$

and for (S_2),

$$\begin{aligned} E_2 &= \left(\frac{\partial R_2}{\partial u}\right)^2, \\ G_2 &= \left(\frac{\partial R_2}{\partial v} + \sqrt{G}\right)^2 + \frac{R_2^2}{h_2^2}\, G, \\ F_2 &= \frac{\partial R_2}{\partial u}\left(\frac{\partial R_2}{\partial v} + \sqrt{G}\right). \end{aligned}$$

These formulæ have been obtained by Professor Craig, who also deduced by aid of them the theorems stated by Darboux (l. c. vol. III, p. 122), that the only surfaces whose lines of curvature have their geodesic curvatures constant, are the surfaces of revolution, the cones, the cylinders and their transformations by inversion.

The angles α_1 and α_2 between the lines u and v on the surfaces (S_1) and (S_2) are given by the following formulæ:

$$\cos\alpha_1 = \frac{\frac{\partial R_1}{\partial u} + \sqrt{E}}{\sqrt{E_1}}\,; \quad \sin\alpha_1 = \frac{R_1\sqrt{E}}{h_1\sqrt{E_1}}\,,$$

$$\cos\alpha_2 = \frac{\frac{\partial R_2}{\partial v} + \sqrt{G}}{\sqrt{G_2}}\,; \quad \sin\alpha_2 = \frac{R_2\sqrt{G}}{h_2\sqrt{G_2}}\,.$$

To find the velocities of translations we use the formulæ

$$\xi_\kappa = \sqrt{E_\kappa}\,.\cos m_\kappa, \quad \eta_\kappa = \sqrt{E_\kappa}\,.\sin m_\kappa,$$
$$\xi_1^{(\kappa)} = \sqrt{G_\kappa}\,.\cos n_\kappa\,, \quad \eta_1^{(\kappa)} = \sqrt{G_\kappa}\,.\sin n_\kappa\,,$$

where

$$n_\kappa - m_\kappa = \alpha_\kappa. \qquad (\varkappa = 1,\ 2)$$

Now for (S_1) the axis of x is tangent to the v-line, hence $n_1 = 0$, $m_1 = -\alpha_1$; for (S_2) the axis of y is tangent to the u-line, i. e. $m_2 = \frac{3\pi}{2}$, so that

$$\cos m_1 = \cos\alpha_1, \quad \sin m_1 = -\sin\alpha_1, \quad \cos n_1 = 1, \quad \sin n_1 = 0,$$
$$\cos m_2 = 0, \quad \sin m_2 = -1, \quad \sin n_2 = -\cos\alpha_2, \quad \cos n_2 = \sin\alpha_2.$$

The translations are:

For (T_1),

$$\xi' = \frac{\partial R_1}{\partial u} + \sqrt{E}; \quad \eta' = -\frac{R_1}{h_1}\sqrt{E} \quad ; \quad \zeta' = 0,$$
$$\xi_1' = \frac{\partial R_1}{\partial v} \quad ; \quad \eta_1' = 0 \quad ; \quad \zeta_1' = 0.$$

For (T_2),

$$\xi^{(2)} = 0 \quad ; \quad \eta^{(2)} = -\frac{\partial R_2}{\partial u} \quad ; \quad \zeta^{(2)} = 0,$$
$$\xi_1^{(2)} = \frac{R_2\sqrt{G}}{h_2} \quad ; \quad \eta_1^{(2)} = -\left(\frac{\partial R_2}{\partial u} + \sqrt{G}\right); \quad \zeta_1^{(2)} = 0.$$

§4.

Returning now to the linear elements ds_1 and ds_2, we see that by making in (13)

$$R_1 + s_1 = Q_1, \quad R_2 + s_2 = Q_2,$$

we can write them as follows:

$$\left.\begin{aligned} ds_1'^2 &= dQ_1^2 + \frac{R_1^2}{h_1^2}\, ds_1^2, \\ ds_2'^2 &= dQ_2^2 + \frac{R_2^2}{h_2^2}\, ds_2^2, \end{aligned}\right\} \tag{14}$$

i. e. the lines

$$R_1 + s_1 = \text{const.},$$
$$R_2 + s_2 = \text{const.},$$

are respectively the orthogonal trajectories of the geodesics

$$s_1 = \text{const.},$$
$$s_2 = \text{const.}$$

on the surfaces (S_1) and (S_2).

From the above expressions of $ds_1'^2$ and $ds_2'^2$ we also see that all the surfaces for which either

$$\frac{R_1}{h_1} \text{ or } \frac{R_2}{h_2}$$

is expressed by the same function of $R_1 + s_1$ and s_1, or $R_2 + s_2$ and s_2 have the corresponding focal surfaces applicable to one another.

If, moreover, this ratio is a function of $R_1 + s_1$ or $R_2 + s_2$ only, the corresponding focal surfaces are applicable to surfaces of revolution.

Another interesting case arises when we have, say

$$h_1 = f(R_1).$$

Then taking the expression of the linear element in the form

$$ds_1'^2 = (dR_1 + ds_1)^2 + \left(\frac{R_1}{h_1}\right)^2 ds_1^2$$

we get

$$ds_1'^2 = dR_1^2 + 2dR_1\, ds_1 + \left(1 + \left(\frac{R_1}{h_1}\right)^2\right) ds_1^2,$$

then

$$1 + \left(\frac{R_1}{h_1}\right)^2 = F(R_1),$$

and $$ds_1'^2 = dR_1^2 + 2dR_1\,ds_1 + F(R_1)\,ds_1^2.$$

Taking $F(R_1)$ outside of the parenthesis,

$$ds_1'^2 = F(R_1)\Big[\Big(\frac{dR_1}{\sqrt{F(R_1)}}\Big)^2 + 2\,\frac{1}{\sqrt{F(R_1)}}\cdot\frac{dR_1}{\sqrt{F(R_1)}}\,ds_1 + ds_1^2\Big],$$

and making

$$\frac{dR_1}{\sqrt{F(R_1)}} = dR_1', \text{ i. e. } R_1' = \int\frac{dR_1}{\sqrt{F(R_1)}},$$

we get

$$\begin{aligned} ds_1'^2 &= \Big[\frac{1}{\phi(R_1')}\Big]^2\{dR_1'^2 + 2\phi(R_1')\,ds_1\,dR_1' + ds_1^2\} \\ &= \Big\{\frac{1}{\phi(R_1')}\Big\}^2[ds_1 + (\phi(R_1') + i\sqrt{1-\phi^2(R_1')})\,dR_1'] \\ &\quad\times[ds_1 + (\phi(R_1') - i\sqrt{1-\phi^2(R_1')})\,dR_1']. \end{aligned}$$

Putting now

$$\begin{aligned} ds_1 + (\phi(R_1') + i\sqrt{1-\phi^2(R_1')})\,dR_1' &= d\alpha, \\ ds_1 + (\phi(R_1') - i\sqrt{1-\phi^2(R_1')})\,dR_1' &= d\beta, \end{aligned}$$

we get

$$ds_1'^2 = \psi(\alpha, \beta)\,d\alpha\,d\beta;$$

which shows that α and β are isometric lines of the focal surface (S_1). This family of isometric lines we can find by quadratures. Namely,

$$\alpha = s_1 + \int[\phi(R_1') + i\sqrt{1-\phi^2(R_1')}]\,dR_1',$$

$$\beta = s_1 + \int[\phi(R_1') - i\sqrt{1-\phi^2(R_1')}]\,dR_1'.$$

Thus we see that in this case the determination of isometric lines on the surface (S_1) is reduced to quadratures.

To determine the function $\psi(\alpha, \beta)$ we subtract the expression for $d\beta$ from that of $d\alpha$ and get

$$d\alpha - d\beta = 2i\sqrt{1-\phi^2(R_1')}\,dR_1',$$

and consequently

$$\alpha - \beta = 2i\int\sqrt{1-\phi^2(R_1')}\,dR_1'.$$

This gives R_1' as a function of α and β. Therefore the determination of the function ψ depends on a quadrature.

Again, writing

$$ds_1'^2 = \frac{1}{\phi^2(R_1')}\left[(ds_1 + \phi(R_1')\,dR_1')^2 + dR_1'^2(1-\phi^2(R_1'))\right],$$

and making

$$ds_1 + \phi(R_1')\,dR_1' = du_1,$$
$$(1-\phi^2(R_1'))^{\frac{1}{2}}\,dR_1' = dv_1,$$

we have

$$ds_1'^2 = \frac{1}{V_1}\left[du_1^2 + dv_1^2\right],$$

where V_1 is a function of v_1 only.

Make

$$\frac{1}{\sqrt{V_1}}\,dv_1 = dv_{11},$$

then

$$ds_1'^2 = dv_{11}^2 + \frac{1}{V_{11}}\,du_1^2.$$

Hence, the lines $u_1 =$ const. and $v_{11} =$ const. constitute also a system of isometric lines, and the focal surface (S_1) is applicable to a surface of revolution. The curves u_1 and v_1 can be obtained by quadratures

$$u_1 = s_1 + \int \phi(R_1')\,dR_1',$$
$$v_1 = \int (1-\phi^2(R_1'))^{\frac{1}{2}}\,dR_1'.$$

The peculiar character of the expression

$$ds_1'^2 = dR_1^2 + 2dR_1\,ds_1 + \left(1+\left(\frac{R_1}{h_1}\right)^2\right)ds_1^2,$$

in case

$$h_1 = f(R_1),$$

namely, that the coefficients of dR_1^2, $dR_1\,ds_1$, ds_1^2 depend only on R_1, gives us a chance to determine all the geodesics of (S_1). (See Darboux, l. c. vol. III, p. 2). We can take

$$\theta = cs_1 + \phi_1(R_1),$$

where

$$\phi_1(R_1) = \int \frac{c + \frac{R_1}{h_1}.\sqrt{1+\frac{R_1^2}{h_1^2}-c^2}}{1+\frac{R_1^2}{h_1^2}}\,dR_1,$$

$\theta =$ const. represents the orthogonal trajectories of the geodesic lines

$$\frac{\partial\theta}{\partial c} = c'.$$

This gives us

$$c' = s_1 + \int \left[\frac{1 - \dfrac{R_1}{h_1\sqrt{1 + \dfrac{R_1^2}{h_1^2} - c^2}}}{1 + \dfrac{R_1^2}{h_1^2}} \right] dR_1,$$

and thus the determination of the geodesic lines on (S_1) is reduced to a quadrature.

Since $V_1 = \phi^2(R_1')$ depends exclusively on the form of the relation $h_1 = f(R_1)$, we conclude that if we have a series of surfaces (Σ) for which h_1 is the same function of R_1, the focal surfaces (S_1) will be applicable to one another and to the same surface of revolution.

Another interesting case is when

$$h_1 = f(s_1),$$

i. e. h_1 is constant along the line $s_1 = \text{const}$. Then the focal surface is a developable surface. Because

$$ds_1'^2 = ds_1^2 + dR_1^2 + 2dR_1\, ds_1 + \frac{R_1^2}{[f(s_1)]^2}\, ds_1^2,$$

or

$$ds_1'^2 = (ds_1 + dR_1)^2 + \frac{R_1^2}{[f(s_1)]^2}\, ds_1^2.$$

Making

$$ds_1 + dR_1 = dv_1,$$

$$\int \frac{ds_1}{f(s_1)} = U_1,$$

we have

$$ds_1'^2 = dv_1^2 + (v_1 - U_1')^2\, dU_1^2,$$

where U_1' is a function of U_1 only. Further on we shall prove that (S_1) can be a developable surface only when the u-line is a plane curve. Hence we conclude that h_1 can be a function of the arc of the line of curvature only when this line is a plane curve. As an example we take a "surface moulure" defined by the equations

$$x = aU \cos\frac{v}{a} + \int V \sin\frac{v}{a}\, dv,$$

$$y = aU \sin\frac{v}{a} - \int V \cos\frac{v}{a}\, dv,$$

$$z = \int \sqrt{1 - a^2 U'^2}\, du.$$

Here U and V are respectively functions of u and v only and a is a constant. For the element of length we get

$$ds^2 = du^2 + (U - V)^2\, dv^2$$

and

$$u = \text{const.}, \quad v = \text{const.}$$

are the lines of curvature of this surface. Here

$$E = 1, \quad F = 0, \quad G = (U - V)^2,$$

$$\frac{1}{R_1} = -\frac{1}{2G\sqrt{E}}\,\frac{\partial G}{\partial u} = -\frac{U'}{U - V},$$

$$\frac{1}{R_2} = 0, \quad \rho_1 = \frac{E}{L} = -\frac{\sqrt{1 - U'^2}}{U''},$$

$$\rho_2 = \frac{G}{N} = \frac{(U - V)^2}{\sqrt{1 - U'^2}},$$

where

$$L = -\frac{U''}{\sqrt{1 - U'^2}}, \quad N = \sqrt{1 - U'^2}, \quad M = 0$$

are the 2nd fundamental coefficients of Gauss. Hence

$$\frac{1}{h_1^2} = \frac{1}{\rho_1^2} = \frac{U''^2}{1 - U'^2} = f(u),$$

and since $s_1 = u$, we have

$$h_1^2 = f(s_1).$$

The focal surface will be a developable surface. Its linear element is

$$ds_1^2 = [d(u + R_1)]^2 + \frac{R_1^2 U''^2}{1 - U'^2} \cdot du^2,$$

or making

$$u + R_1 = t, \quad \int \frac{U''}{\sqrt{1 - U'^2}}\, du = \text{arc sin } U' = Z,$$

we have

$$ds_1^2 = dt^2 + (t - F(Z))^2\, dZ.$$

To finish this subject, we prove a theorem suggested by a note of T. Caronnet.*

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* Th. Caronnet, C. R. 1892, "Sur les centres de courbure géodésique." The following proof is identical with the one given by Caronnet.

We are going to prove that if there exists a relation

$$h_1 = f(R_1),$$

then the lines connecting the centers of geodesic curvature of the u-line with the corresponding centers of principal curvature of the v-line, form a congruence of rays normal to a surface.

To prove this we consider our surface (Σ) as generated by the motion of a trihedron formed by the tangent to the u-line, its normal and binormal. Consider the points $(R_1, 0, 0)$ and $(0, h_1, 0)$ and imagine a straight line through those two points. Any point on this line is represented by the following coordinates:

$$\begin{aligned} x_1 &= R_1 - \lambda \cdot \frac{R_1}{\sqrt{R_1^2 + h_1^2}}, \\ y_1 &= \lambda \cdot \frac{h_1}{\sqrt{R_1^2 + h_1^2}}, \\ z_1 &= 0, \end{aligned}$$

λ being the distance from $(R_1, 0, 0)$ to (x_1, y_1, z_1).

Consider the displacements on the axes x, y, z of the point (x, y, z); they are

$$\begin{aligned} &dx_1 + \sqrt{E}du + z_1(qdu + q_1dv) - y_1(rdu + r_1dv), \\ &dy_1 + \sqrt{G}\,dv + x_1(rdu + r_1dv) - z_1(pdu + p_1dv), \\ &dz_1 + \qquad\quad + y_1(pdu + p_1dv) - x_1(qdu + q_1dv). \end{aligned}$$

In order that the point (x_1, y_1, z_1) describe a surface to which the line abovementioned be normal, the angle between the displacement of (x_1, y_1, z_1) and this line must be a right angle, whatever be du and dv. The direction cosines of our line are

$$\frac{R_1}{\sqrt{R_1^2 + h_1^2}}, \quad -\frac{h_1}{\sqrt{R_1^2 + h_1^2}}, \quad 0.$$

Consequently

$$\begin{aligned} &\frac{R_1}{\sqrt{R_1^2 + h_1^2}}\left[dx_1 + \sqrt{E}\,du + z_1(qdu + q_1\,dv) - y_1(rdu + r_1\,dv)\right], \\ &-\frac{h_1}{\sqrt{R_1^2 + h_1^2}}\left[dy_1 + \sqrt{G}\,dv + x_1(rdu + r_1\,dv) - z_1(pdu + p_1\,dv)\right] = 0, \end{aligned}$$

and since this equation holds for any du and dv, we get

$$R_1\,dx - h_1\,dy = 0. \tag{15}$$

But

$$dx_1 = dR_1 - \lambda\,.\,\frac{h_1^2\,dR_1 - R_1 h_1\,dh_1}{(R_1^2 + h_1^2)^{\frac{3}{2}}} - \frac{R_1\,d\lambda}{\sqrt{R_1^2 + h_1^2}},$$

$$dy_1 = \lambda\,.\,\frac{R_1^2\,dh_1 - h_1 R_1\,dR_1}{(R_1^2 + h_1^2)^{\frac{3}{2}}} + h_1\,.\,\frac{d\lambda}{\sqrt{R_1^2 + h_1^2}},$$

and the above condition becomes after a few reductions

$$\tfrac{1}{2}\,dR_1^2 - \sqrt{R_1^2 + h_1^2}\,.\,d\lambda = 0$$

and

$$d\lambda = \frac{dR_1^2}{2\sqrt{R_1^2 + h_1^2}}.$$

Thus we see that λ can be determined only when R_1 is a function of h_1. In this case we can write

$$\lambda = \tfrac{1}{2}\int \frac{dR_1^2}{\sqrt{R_1^2 + f(R_1)}},$$

and this determines the point which describes the required surface. Conversely, if the above-mentioned lines form a congruence of normals to a surface, the equation (15) holds and

$$R_1 = f(h_1).$$

Returning to the expression

$$ds_1'^2 = dq_1^2 + \frac{R_1^2}{h_1^2}\,ds_1^2,$$

we see that the linear element of the focal surface depends only on the ratio $\frac{R_1}{h_1}$. If for example

$$\frac{R_1}{h_1} = Q_1 - S_1,$$

where Q_1 and S_1 are functions of q_1 and s_1 respectively, the generated surface (S_1) is a "surface moulure" and

$$q_1 = \text{const.} \quad \text{and} \quad s_1 = \text{const.}$$

are the lines of curvature of (S_1).

Again, if

$$\frac{R_1}{h_1} = m,$$

where m is a constant, we obtain a developable surface, and as previously remarked, this may take place only when the corresponding lines of curvature are plane curves.

If we suppose

$$\frac{R_1}{h_1} = e^{2q_1} \text{ or } \frac{e^{q_1} \pm e^{-q_1}}{2},$$

we shall have for (S_1) a surface of revolution whose total curvature is constant.

§5.

We pass now to the elements of the 2nd order of the focal surfaces (S_1) and (S_2). We must find the rotations p', q', p_1', q_1', r_1' of (S_1) and p'', q'', r'', p_1'', q_1'', r_1'' of (S_2). We know that if a, b, c and the accented letters are the direction cosines of the movable axes with respect to the fixed axes, according to the table

	X	Y	Z
x	a	a'	a''
y	b	b'	b''
z	c	c'	c''

(X, Y, Z are the fixed and x, y, z the movable axes), then the rotations are expressed by the formulæ

$$p = \Sigma c \frac{db}{du}, \quad q = -\Sigma c \frac{da}{du}, \quad r = \Sigma b \frac{da}{du},$$

$$p_1 = \Sigma c \frac{db}{dv}, \quad q_1 = -\Sigma c \frac{da}{dv}, \quad r_1 = \Sigma b \frac{da}{dv}.$$

We have already remarked that in order to pass from (T) to (T_1) or (T_2) we must rotate the trihedron (T) around the axes x and y through the angles ϖ_1 and ϖ_2 respectively.

The following is a table of direction cosines of the movable axes x, y, z

with the fixed axes and with the axes x', y', z', x'', y'', z'' of the trihedrons (T_1) and (T_2):

	X	Y	Z	x'	y'	z'	x''	y''	z''
x	a	a'	a''	1	0	0	$\sin\varpi_2$	0	$-\cos\varpi_2$
y	b	b'	b''	0	$\sin\varpi_1$	$\cos\varpi_1$	0	1	0
z	c	c'	c''	0	$\cos\varpi_1$	$-\sin\varpi_1$	$\cos\varpi_2$	0	$\sin\varpi_2$

Hence we can write

$$\begin{aligned}
&\cos(Xx') = a, \quad \cos(Yx') = a', \quad \cos(Zx') = a'',\\
&\cos(Xy') = b\sin\varpi_1 + c\cos\varpi_1,\\
&\cos(Yy') = b'\sin\varpi_1 + c'\cos\varpi_1,\\
&\cos(Zy') = b''\sin\varpi_1 + c''\cos\varpi_1,\\
&\cos(Xz') = b\cos\varpi_1 - c\sin\varpi_1,\\
&\cos(Yz') = b'\cos\varpi_1 - c'\sin\varpi_1,\\
&\cos(Zz') = b''\cos\varpi_1 - c''\sin\varpi_1,\\
&\cos(Xx'') = a\sin\varpi_2 + c\cos\varpi_2,\\
&\cos(Yx'') = a'\sin\varpi_2 + c'\cos\varpi_2,\\
&\cos(Zx'') = a''\sin\varpi_2 + c''\cos\varpi_2,\\
&\cos(Xy'') = b, \quad \cos(Yy'') = b', \quad \cos(Zy'') = b'',\\
&\cos(Xz'') = -a\cos\varpi_2 + c\sin\varpi_2,\\
&\cos(Yz'') = -a'\cos\varpi_2 + c'\sin\varpi_2,\\
&\cos(Zz'') = -a''\cos\varpi_2 + c''\sin\varpi_2.
\end{aligned}$$

We can now write the expressions for the rotations. Namely,

$$\begin{aligned}
p' &= \Sigma c_1 \frac{db_1}{du} = p + \frac{\partial\varpi_1}{\partial u},\\
p_1' &= p_1 + \frac{\partial\varpi_1}{\partial v},\\
q' &= -\Sigma c_1 \frac{da_1}{du} = -(r\cos\varpi_1 + q\sin\varpi_1),\\
q_1' &= -(r_1\cos\varpi_1 + q_1\sin\varpi_1),\\
r' &= \Sigma b_1 \frac{da_1}{du} = r\sin\varpi_1 - q\cos\varpi_1,\\
r_1' &= r_1\sin\varpi_1 - q_1\cos\varpi_1.
\end{aligned}$$

And similarly,

$$
\begin{aligned}
p'' &= r \cos \varpi_2 + p \sin \varpi_2, \\
p_1'' &= r_1 \cos \varpi_2 + p_1 \sin \varpi_2, \\
q'' &= q + \frac{\partial \varpi_2}{\partial u}, \\
q_1'' &= q_1 + \frac{\partial \varpi_2}{\partial v}, \\
r'' &= r \sin \varpi_2 - p \cos \varpi_2, \\
r_2'' &= r_1 \sin \varpi_2 - p_1 \cos \varpi_2.
\end{aligned}
$$

Introducing in these formulæ the values of p, q, r, p_1, q_1, r_1, $\cos \varpi_1$, $\sin \varpi_1$, $\cos \varpi_2$, $\sin \varpi_2$, we obtain the following table:

$$
\begin{aligned}
p' &= \frac{\partial \varpi_1}{\partial u} = \frac{\sqrt{E}}{\tau_1}, & q' &= 0, \\
p_1' &= p_1 + \frac{\partial \varpi_1}{\partial v}, & q_1' &= \frac{h_1 \sqrt{G}}{R_1 \rho_1}, \\
r' &= \frac{\sqrt{E}}{h_1}, & r_1' &= -\frac{h_1 \sqrt{G}}{R_1 R_2}, \\
p'' &= -\frac{h_2 \sqrt{E}}{\rho_2 R_2}, & p_1'' &= 0, \\
q'' &= q + \frac{\partial \varpi_2}{\partial u}, & q_1'' &= \frac{\partial \varpi_2}{\partial v} = \frac{\sqrt{G}}{\tau_2}, \\
r'' &= -\frac{h_2 \sqrt{E}}{R_1 R_2}, & r_1'' &= -\frac{\sqrt{G}}{h_2}.
\end{aligned}
$$

Still following Darboux's notations according to which the 2nd fundamental coefficients of Gauss, L, M, N are given by

$$
L = \frac{D}{\Delta}, \quad M = \frac{D'}{\Delta}, \quad N = \frac{D''}{\Delta},
$$

where the D's are given by the equations

$$
\left.
\begin{aligned}
\Delta^2 p &= \xi D' - \xi_1 D; & \Delta^2 q &= \eta D' - \eta_1 D, \\
\Delta^2 p_1 &= \xi D'' - \xi_1 D'; & \Delta^2 q_1 &= \eta D'' - \eta_1 D',
\end{aligned}
\right\} \qquad (16)
$$

and

$$
\Delta = \eta_1 \xi - \eta \xi_1, \qquad (17)
$$

we get, by applying successively the above formulæ to the surfaces (S_1) and (S_2),

$$\Delta' = -\eta' \xi_1', \quad \Delta'' = -\eta'' \xi_1''. \tag{18}$$

Then for (S_1): $q_1' = 0$, $\eta_1' = 0$; hence

$$D_1' = 0,$$

and for (S_2): $p_1'' = 0$, $\xi'' = 0$; hence

$$D_2' = 0.$$

That is,

$$M_1 = 0, \quad M_2 = 0,$$

and consequently the lines u and v on the surfaces (S_1) and (S_2) are conjugate. This is a well-known property of the focal surfaces.

Now

$$L_1 = -\frac{D_1}{\Delta_1} = -\frac{R_1}{h_1}\sqrt{E}\,.\,\frac{\partial \varpi_1}{\partial u},$$

or

$$L_1 = -\frac{R_1 E}{h_1 \tau_1}$$

and

$$N_1 = \frac{D_2}{\Delta_1} = -\frac{\partial R_1}{\partial v} \cdot \frac{h_1 \sqrt{G}}{R_1 \rho_1}.$$

Similarly

$$L_2 = -\frac{\sqrt{E}\,.\,h_2}{R_2 \rho_2} \cdot \frac{\partial R_2}{\partial u}, \quad N_2 = -\frac{R_2 G}{h_2 \tau_2}. \tag{19}$$

The measures of total curvature of (S_1) and (S_2) will be

for (S_1)

$$\frac{1}{\rho_1' \rho_2'} = \frac{1}{\frac{\partial R_1}{\partial v}} \cdot \frac{h_1^2 \sqrt{G}}{\tau_1 R_1^2 \rho_1};$$

for (S_2),

$$\frac{1}{\rho_1'' \rho_2''} = \frac{1}{\frac{\partial R_2}{\partial u}} \cdot \frac{h_2^2 \sqrt{E}}{\tau_2\,.\,R_2^2 \rho_2}.$$

In order that any of these expressions may become zero, i. e. in order that any of ρ_1', ρ_2', ρ_1'', ρ_2'' may become infinitely large and the focal surface may become a developable surface, we must have any one of the following equations:

$$\tau_1 = \infty, \quad \tau_2 = \infty, \quad R_1 = \infty, \quad R_2 = \infty, \quad \rho_1 = \infty, \quad \rho_2 = \infty.$$

All of these equations show that the lines of curvature of (Σ) in one system or in both systems must be plane curves. And conversely, if (S_1) or (S_2) be developable surfaces, the corresponding lines of curvature must be plane curves.

The expressions for the rotations p_1' and q'' give us a means of finding the derivatives

$$\frac{\partial \varpi_1}{\partial v}, \quad \frac{\partial \varpi_2}{\partial u}.$$

From
$$\Delta_1^2 p_1' = \xi' D_1'', \quad \Delta_{11}^2 q'' = -\eta_1'' D_{11}$$

we have
$$p_1' = \frac{D_1''\xi'}{\Delta_1^2} = -\frac{h_1^2}{R_1^2\rho_1}\left(\frac{\partial R_1}{\partial u} + \sqrt{E}\right) \cdot \sqrt{\frac{G}{E}}$$

and
$$\frac{\partial \varpi_1}{\partial v} = -\left[\frac{1}{h_2} + \frac{h_1^2}{\sqrt{E} \,.\, R_1^2\rho_1}\left(\frac{\partial R_1}{\partial u} + \sqrt{E}\right)\right] \cdot \sqrt{G},$$

or remarking that the derivatives

$$\frac{\partial}{\sqrt{G} \,.\, \partial v} \text{ and } \frac{\partial}{\sqrt{E} \,.\, \partial u}$$

represent the derivatives with respect to the displacements along the lines of curvature ds_2 and ds_1, we can write

$$\frac{\partial \varpi_1}{\partial s_2} = -\left[\frac{1}{h_2} + \frac{h_1^2}{\rho_1 R_1^2}\left(\frac{\partial R_1}{\partial s_1} + 1\right)\right].$$

In the same way we have

$$q'' = -\left[\frac{\partial R_2}{\partial v} + \sqrt{G}\right] \cdot \frac{h_2}{R_2^2\rho_2} \cdot \sqrt{\frac{E}{G}}$$

and
$$\frac{\partial \varpi_2}{\partial s_1} = -\left[\frac{1}{h_1} + \frac{h_2^2}{\rho_2 R_2^2}\left(\frac{\partial R_2}{\partial s_2} + 1\right)\right].$$

§6.

The equations of the asymptotic lines are:

for (S_1),
$$\frac{R_1^2}{h_1^2} \,.\, \sqrt{E} \,.\, \frac{\partial \varpi_1}{\partial u} \,.\, du^2 + \frac{1}{\rho_1} \,.\, \frac{\partial R_1}{\partial v} \,.\, \sqrt{G} \,.\, dv^2 = 0\,;$$

for (S_2),
$$\frac{R_2^2}{h_2^2} \,.\, \sqrt{G} \,.\, \frac{\partial \varpi_2}{\partial v} \,.\, dv^2 + \frac{1}{\rho_2} \,.\, \frac{\partial R_2}{\partial u} \,.\, \sqrt{E} \,.\, du^2 = 0.$$

By inspection we find from these equations that in order that the lines of curvature of (Σ) correspond to the asymptotic lines of (S_1) and (S_2), we must have simultaneously

$$\frac{\partial \varpi_1}{\partial u} = 0 \quad , \quad \frac{\partial \varpi_2}{\partial v} = 0 \quad , \tag{20}$$

or

$$\varpi_1 = f(v), \quad \varpi_2 = f_1(u),$$

i. e. the angles between the osculating planes of the u and the v lines and the tangent planes to (Σ) must be constant. The converse is also correct. But the equations (20) give

$$\tau_1 = \infty, \quad \tau_2 = \infty,$$

i. e. that the lines of curvature of (Σ) in both systems must be plane.

We know all the surfaces having lines of curvature plane in both systems. They are the envelopes of the plane

$$\alpha x - \beta y + (\lambda \sqrt{1 - x^2} - \sqrt{\lambda^2 - 1} \cdot \sqrt{1 + \beta^2}) z$$
$$= f(\alpha) - f(\beta),$$

where α and β are parameters of the lines of curvature. Hence we know all the surfaces for which the lines of curvature correspond to the asymptotic lines of the focal surfaces (S_1) and (S_2).

The equations of the asymptotic lines of (Σ) are

$$\frac{E}{\rho_1} du^2 + \frac{G}{\rho_2} dv^2 = 0.$$

In order that the asymptotic lines of (Σ) may correspond to those of (S_1) we must have

$$\left.\begin{aligned} \frac{E}{\rho_1} &= m \cdot \frac{R_1^2}{h_1^2} \cdot \sqrt{E} \cdot \frac{\sqrt{E}}{\tau_1}, \\ \frac{G}{\rho_2} &= m \cdot \frac{1}{\rho_1} \cdot \frac{\partial R_1}{\partial v} \cdot \sqrt{G}. \end{aligned}\right\} \tag{21}$$

Now we have seen that for (S_1)

$$\frac{1}{\rho_1' \rho_2'} = \frac{1}{\frac{\partial R_1}{\partial v}} \cdot \frac{h_1^2 \sqrt{G}}{\tau_1 \cdot R_1^2 \rho_1},$$

and from the equations (21),

$$\frac{1}{\rho_1 \rho_2} = \frac{1}{\rho_1^3} \cdot \frac{\partial R_1}{\partial v} \cdot \frac{h_1^2}{R_1^2} \cdot \tau_1 \cdot \frac{1}{\sqrt{G}},$$

and finally,

$$\rho_1'\rho_2'\rho_1\rho_2=\left(\frac{R_1\rho_1}{h_1}\right)^4.$$

Calling

$$V\equiv\frac{\pi}{2}-\varpi_1,$$

we get

$$h_1'h_2'h_1h_2=\left(\frac{R_1}{\sin V}\right)^4.$$

This is a particular case of a theorem given by E. Cosserat* and A. Demoulin.† It follows at once that if we have the above relation between the radii of principal curvatures of (Σ) and (S_1), their asymptotic lines correspond. For then

$$\rho_1'\rho_2'=\frac{1}{\rho_1\rho_2}\cdot\frac{R_1^4\rho_1^4}{h_1^4},$$

and since for (S_1) we have

$$\frac{1}{\rho_1'\rho_2'}=\frac{1}{\frac{\partial R_1}{\partial v}}\cdot\frac{h_1^2\sqrt{G}}{\tau_1\,.\,R_1^2\rho_1},$$

the above equation becomes

$$\frac{1}{\frac{1}{\frac{\partial R_1}{\partial v}}\cdot\frac{h_1^2\sqrt{G}}{\tau\,.\,R_1^2\rho_1}}=\frac{1}{\rho_1\rho_2}\cdot\frac{R_1^4\rho_1^4}{h_1^4},$$

and hence

$$\frac{E}{\rho_1}:\frac{G}{\rho_2}=\frac{R_1^2}{h_1^2}\,\frac{E}{\tau_1}:\frac{1}{h_1}\,\frac{\partial R_1}{\partial v}\sqrt{G},$$

which shows that the two equations of the asymptotic lines for (Σ) and (S_1) are identical.

So that we can say: if the product of four radii of principal curvatures of (Σ) and (S_1) is equal to the fourth power of the fourth proportional to R_1, ρ_1 and h_1, the asymptotic lines on both surfaces correspond. And if we desire to have the asymptotic lines of (S_2) correspond to those of (S_2), we must have

$$\frac{R_1^2}{h_1^2}\cdot\sqrt{E}\cdot\frac{\sqrt{E}}{\tau_1}=m\cdot\frac{1}{\rho_2}\cdot\frac{\partial R_2}{\partial u}\cdot\sqrt{E},$$

$$\frac{1}{\rho_1}\cdot\frac{\partial R_1}{\partial v}\cdot\sqrt{G}=m\cdot\frac{R_2^2}{h_2^2}\cdot\sqrt{G}\cdot\frac{\sqrt{G}}{\tau_2}.$$

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*E. Cosserat, C. R., vol. 118, 1894, "Sur des congruences rectilignes et sur le problème de Ribaucour."

†A. Demoulin, C. R., 1894, "Sur une propriété métrique commune à trois classes particulières de congruences rectilignes."

Now we had

$$\frac{1}{\rho_1' \rho_2'} = \frac{1}{\frac{\partial R_1}{\partial v}} \cdot \frac{h_1^2 \sqrt{G}}{\tau_1 . R_1^2 \rho_1},$$

$$\frac{1}{\rho_2'' \rho_1''} = \frac{1}{\frac{\partial R_2}{\partial u}} \cdot \frac{h_2^2 \sqrt{E}}{\tau_2 R_2^2 \rho_2},$$

or

$$\rho_1' \rho_2' \rho_1'' \rho_2'' = \frac{\tau_1 \tau_2 \rho_1 \rho_2 R_1^2 R_2^2}{h_1^2 h_2^2 \sqrt{EG}} \cdot \frac{\partial R_1}{\partial v} \cdot \frac{\partial R_2}{\partial u}.$$

From the first two equations

$$\frac{\partial R_1}{\partial u} \cdot \frac{\partial R_2}{\partial v} \cdot \frac{\tau_1 \tau_2}{\sqrt{EG}} = \frac{R_1^2 R_2^2 \rho_1 \rho_2}{h_1^2 h_2^2},$$

hence

$$\rho_1' \rho_2' \rho_1'' \rho_2'' = \frac{R_1^4 R_2^4 \rho_1^2 \rho_2^2}{h_1^4 h_2^4},$$

or if we want to introduce the angles ϖ_1 and ϖ_2 we have

$$\frac{R_1}{h_2} = \frac{1}{\sin \varpi_2}, \quad \frac{R_2}{h_1} = \frac{1}{\sin \varpi_1}$$

and

$$\rho_1' \rho_2' \rho_1'' \rho_2'' = \left(\frac{\rho_1 \rho_2}{\sin^2 \varpi_1 \sin^2 \varpi_2} \right)^2,$$

or again,

$$\rho_1' \rho_2' \rho_1'' \rho_2'' = \left(\frac{2R_1 . 2R_2}{\sin 2\varpi_1 \sin 2\varpi_2} \right)^2.$$

In a way identical to that on page 126, we can prove that this relation is also sufficient for or characteristic of the correspondence of the asymptotic lines on (S_1) and (S_2).

From the relations

$$\rho_1' \rho_2' \rho_1 \rho_2 = \left(\frac{R_1 \rho_1}{h_1} \right)^4$$

and

$$\rho_1' \rho_2' \rho_1'' \rho_2'' = \frac{R_1^4 R_2^4 \rho_1^2 \rho_2^2}{h_1^4 h_2^4},$$

we conclude that if there exists a correspondence of the asymptotic lines on (Σ) and (S_1) or (S_1) and (S_2), then at corresponding points both surfaces are either convex or of opposite curvature. For instance, if $\rho_1' > 0$, $\rho_2' > 0$, then $\rho_1^{(\kappa)}$ and

$\rho_2^{(\kappa)}$ must be either both $>$ or both < 0, $(\varkappa = 0, 2)$. If on the contrary $\rho_1' > 0$, $\rho_2' < 0$, then $\rho_1^{(\kappa)} \gtrless 0$ and $\rho_2^{(\kappa)} \lessgtr 0$.

Again for such surfaces, if (Σ) has its total curvature constant, (S_1) can be of the same kind only if

$$\frac{R_1 \rho_1}{h_1} = \text{const.}$$

Since the equations of the asymptotic lines and the equations connecting any two conjugate directions du and δu, dv and δv, contain the same coefficients, we may say that when the above relations, connecting the four radii of principal curvature, exist, then the conjugate directions of (Σ) and (S_1) or of (S_1) and (S_2) correspond.

§7.

The equations of the lines of curvature we write in two different ways, viz. For (S_1) (calling ρ_1^0, ρ_2^0 the radii of principal curvature of (S_1) and (S_2),

$$\left.\begin{aligned} \left(\frac{\partial R_1}{\partial u} + \sqrt{E}\right) du + \frac{\partial R_1}{\partial v} dv + \rho_1^0 \frac{h_1 \sqrt{G}}{R_1 \rho_1} dv &= 0, \\ \frac{R_1}{h_1} \sqrt{E}\, du + \rho_1^0 \left(\frac{\partial \varpi_1}{\partial u} du + p_1 dv + \frac{\partial \varpi_1}{\partial v} dv\right) &= 0. \end{aligned}\right\} \tag{22}$$

For (S_2),

$$\left.\begin{aligned} \left(\frac{\partial R_2}{\partial v} + \sqrt{G}\right) dv + \frac{\partial R_2}{\partial u} du + \rho_2^0 \frac{h_2 \sqrt{E}}{R_2 \rho_2} du &= 0, \\ \frac{R_2}{h_2} \sqrt{G} . dv + \rho_2^0 \left(\frac{\partial \varpi_2}{\partial v} . dv + q du + \frac{\partial \varpi_2}{\partial u} du\right) &= 0. \end{aligned}\right\} \tag{23}$$

Eliminating from these four equations ρ_1^0 and ρ_2^0, we obtain the differential equations of the lines of curvature of (S_1) and (S_2).

For (S_1),

$$\begin{aligned} &\left(\frac{\partial R_1}{\partial u} + \sqrt{E}\right) \cdot \frac{\rho_1}{\tau_1} . \sqrt{E}\, du^2 + \left(\frac{\partial R_1}{\partial v} \cdot \frac{\rho_1}{\tau_1} . \sqrt{E} - \sqrt{EG}\right. \\ &\qquad \left. - \sqrt{\frac{G}{E}} \cdot \frac{h_1^2}{R_1^2} \left(\frac{\partial R_1}{\partial u} + \sqrt{E}\right)^2\right) du\, dv \\ &\qquad - \frac{h_1^2}{R_1^2} \left(\frac{\partial R_1}{\partial u} + \sqrt{E}\right) . \sqrt{\frac{G}{E}} \cdot \frac{\partial R_1}{\partial v} dv^2 = 0, \end{aligned}$$

and for (S_2),

$$\begin{aligned}\left(\frac{\partial R_2}{\partial v}+\sqrt{G}\right)\frac{\rho_2}{\tau_2}\sqrt{G}\,.\,dv^2+\Bigg(\frac{\partial R_2}{\partial u}\cdot\frac{\rho_2}{\tau_2}\cdot\sqrt{G}-\sqrt{EG}\\ -\sqrt{\frac{E}{G}}\cdot\frac{h_2^2}{R_2^2}\cdot\left(\frac{\partial R_2}{\partial v}+\sqrt{G}\right)^2\Bigg)du\,dv\\ -\frac{h_2^2}{R_2^2}\left(\frac{\partial R_2}{\partial v}+\sqrt{G}\right)\cdot\sqrt{\frac{E}{G}}\cdot\frac{\partial R_2}{\partial u}du^2=0.\end{aligned}$$

From these equations we can see that if we want u and v to remain parameters of the lines of curvature on both focal surfaces (S_1) and (S_2), we must have simultaneously

$$\frac{\partial R_1}{\partial u}+\sqrt{E}=0,\quad \frac{\partial R_2}{\partial v}+\sqrt{G}=0.$$

The first equation gives

$$\frac{\partial R_1}{\partial u}=-\sqrt{E},$$

and since

$$R_1=-\frac{\sqrt{G}}{r_1},$$

$$\frac{\partial R_1}{\partial u}=-\frac{1}{r_1}\cdot\frac{\partial\sqrt{G}}{\partial u}+\frac{\sqrt{G}}{r_1^2}\cdot\frac{\partial r_1}{\partial u}=-\sqrt{E}.$$

Again,

$$r_1=\frac{1}{\sqrt{E}}\cdot\frac{\partial\sqrt{G}}{\partial u},\quad \sqrt{E}=\frac{1}{r_1}\cdot\frac{\partial\sqrt{G}}{\partial u},$$

hence

$$\frac{\sqrt{G}}{r_1^2}\cdot\frac{\partial r_1}{\partial u}=0,$$

i. e.

$$\frac{\partial r_1}{\partial u}=0.$$

The second equation gives

$$\frac{\partial R_2}{\partial v}=-\sqrt{G}=\frac{1}{r}\cdot\frac{\partial\sqrt{E}}{\partial v}-\frac{\sqrt{E}}{r^2}\cdot\frac{\partial r}{\partial v},$$

and since

$$r=-\frac{1}{\sqrt{G}}\cdot\frac{\partial\sqrt{E}}{\partial v};\quad \sqrt{G}=-\frac{1}{r}\cdot\frac{\partial\sqrt{E}}{\partial v},$$

we have

$$-\frac{\sqrt{E}}{r^2}\cdot\frac{\partial r}{\partial v}=0,$$

i. e.

$$\frac{\partial r}{\partial v}=0.$$

Hence
$$\frac{\partial r}{\partial v} - \frac{\partial r}{\partial u} = - q p_1 = 0,$$

which requires either $q = 0$ or $p_1 = 0$.

If $q = 0$,
$$\frac{\partial q}{\partial v} = \quad r p_1 = 0 \text{ and } p_1 = 0;$$

if $p_1 = 0$,
$$\frac{\partial p_1}{\partial u} = - q r_1 = 0 \text{ and } q = 0.$$

We excluded from consideration the cases where r or r_1 are equal to zero, for then the surface (Σ) would belong to the class of surfaces enumerated by Darboux (l. c. v. III, p. 122) and the focal surface would degenerate into a line, so that we have
$$p_1 = 0 \;, \quad q = 0 \;,$$
i. e.
$$\rho_1 = \infty, \quad \rho_2 = \infty,$$

and we get a trivial solution of the plane. This theorem has been proved by Professor Craig in a somewhat different way.

The question, can the lines of curvature of (Σ) correspond to those of say (S_1)? has been treated by C. Guichard* in connection with the surfaces of constant total curvature.

Let us eliminate from the equations (22) and (23) $\frac{dv}{du}$; we obtain equations determining radii of principal curvature of (S_1) and (S_2).

For (S_1),
$$\frac{\partial \varpi_1}{\partial u}\;\frac{h_1}{R_1}\;\frac{\sqrt{G}}{\rho_1}\,\rho^2 - \rho\left[\left(p_1 + \frac{\partial \varpi_1}{\partial v}\right)\left(\frac{\partial R_1}{\partial u} + \sqrt{E}\right) - \frac{\sqrt{EG}}{h_1} - \frac{\partial R_1}{\partial v}\cdot\frac{\partial \varpi_1}{\partial u}\right] + \frac{R_1 . \sqrt{E}}{h_1}\cdot\frac{\partial R_1}{\partial v} = 0.$$

For (S_2),
$$\frac{\partial \varpi_2}{\partial v}\;\frac{h_2\sqrt{E}}{R_2\,\rho_2}\,\rho^2 - \rho\left[\left(q + \frac{\partial \varpi_2}{\partial u}\right)\left(\frac{\partial R_2}{\partial v} + \sqrt{G}\right) - \frac{\sqrt{EG}}{h_2} - \frac{\partial R_2}{\partial u}\cdot\frac{\partial \varpi_2}{\partial v}\right] + \frac{R_2\sqrt{G}}{h_2}\cdot\frac{\partial R_2}{\partial u} = 0.$$

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[*] C. Guichard, "Recherches sur les surfaces à courbure total constante," etc. (Ann. de l'École Normale, 1890).

Calling K_1 and K_2 the respective total curvatures of (S_1) and (S_2) and H_1 and H_2 their mean curvatures, we have

$$H_1 = \frac{h_1\left[\left(p_1 + \frac{\partial \varpi_1}{\partial v}\right)\left(\frac{\partial R_1}{\partial u} + \sqrt{E}\right) - \frac{\sqrt{EG}}{\rho_1} - \frac{\partial R_1}{\partial v}\cdot\frac{\partial \varpi_1}{\partial u}\right]}{R_1\sqrt{E}\,.\,\frac{\partial R_1}{\partial v}},$$

$$K_1 = \frac{h_1^2\sqrt{G}\,.\,\frac{\partial \varpi_1}{\partial u}}{R_1^2\rho_1\frac{\partial R_1}{\partial v}\,.\,\sqrt{E}},$$

$$H_2 = \frac{h_2\left[\left(q + \frac{\partial \varpi_2}{\partial u}\right)\left(\frac{\partial R_2}{\partial v} + \sqrt{G}\right) - \frac{\sqrt{EG}}{\rho_2} - \frac{\partial R_2}{\partial u}\cdot\frac{\partial \varpi_2}{\partial v}\right]}{R_2\sqrt{G}\,.\,\frac{\partial R_1}{\partial v}},$$

$$K_2 = \frac{h_2^2\sqrt{E}\,.\,\frac{\partial \varpi_2}{\partial v}}{R_2^2\,.\,\rho_2\,.\,\frac{\partial R_2}{\partial u}\sqrt{G}}.$$

In case the lines of curvature of (Σ) correspond to those of (S_1), we have

$$\frac{\partial R_1}{\partial u} + \sqrt{E} = 0,$$

$$H_1 = -\frac{\sqrt{EG} + \frac{\partial R_1}{\partial v}\cdot\frac{\partial \varpi_1}{\partial u}}{R_1\,.\sqrt{E}\,.\,\frac{\partial R_1}{\partial v}}; \quad K_1 = \frac{h_1^2\sqrt{G}\,.\,\frac{\partial \varpi_1}{\partial u}}{R_1^2\rho_1\frac{\partial R_1}{\partial v}\,.\sqrt{E}}.$$

In this case we can write very simple expressions for the radii of principal curvatures. Since u and v are parameters of lines of curvature on (S_1), we have

$$\rho_1' = \frac{E_1}{L_1} = -\frac{R_1^2 E}{h_1^2}\cdot\frac{h_1\tau_1}{ER_1} = -\frac{R_1\tau_1}{h_1},$$

$$\rho_2' = \frac{G_1}{N_1} = -\frac{\left(\frac{\partial R_1}{\partial v}\right)^2\,.\,R_1\rho_1}{h_1\,.\sqrt{G}\,.\,\frac{\partial R_1}{\partial v}} = -\frac{R_1\rho_1}{h_1\sqrt{G}}\cdot\frac{\partial R_1}{\partial v}.$$

Thus we see that ρ_1' is the fourth proportional to R_1, τ_1, h_1. This property furthermore is characteristic of the surfaces. For suppose that one of the radii of curvature of the focal surface (S_1), of the congruence of tangents to the line of curvature $v =$ const. of the surface (Σ), is the fourth proportional to R_1, τ_1, h_1. Then for the second radius of curvature we shall have

$$\rho_2' = -\frac{R_1 \rho_1}{h_1 \sqrt{G}} \cdot \frac{\partial R_1}{\partial v},$$

and this requires

$$\left(p_1 + \frac{\partial \varpi_1}{\partial v}\right)\left(\frac{\partial R_1}{\partial u} + \sqrt{E}\right) = 0,$$

or

$$\frac{h_1^2 \sqrt{G}}{\sqrt{E} \,.\, R_1^2 \rho_1} \left(\frac{\partial R_1}{\partial u} + \sqrt{E}\right)^2 = 0,$$

or

$$\left(\frac{\partial R_1}{\partial u} + \sqrt{E}\right) = 0. \qquad \text{Q. E. D.}$$

Therefore the necessary and sufficient condition that the lines of curvature on (Σ) and (S_1) correspond is that

$$\rho_1' = \frac{R_1 \tau_1}{h_1}.$$

Consider now two surfaces (S_1) and (S_2) such that the lines of curvature of one correspond to those of the other. We must have

$$\left(\frac{\partial R_1}{\partial u} + \sqrt{E}\right) \cdot \frac{\rho_1}{\tau_1} \,.\, \sqrt{E} = -m \,.\, \frac{h_2^2}{R_2^2} \cdot \left(\frac{\partial R_2}{\partial v} + \sqrt{G}\right) \cdot \sqrt{\frac{E}{G}} \cdot \frac{\partial R_2}{\partial u},$$

$$-\left(\frac{\partial R_1}{\partial u} + \sqrt{E}\right) \cdot \frac{h_1^2}{R_1^2} \cdot \sqrt{\frac{G}{E}} \cdot \frac{\partial R_1}{\partial v} = m \,.\left(\frac{\partial R_2}{\partial v} + \sqrt{G}\right) \cdot \frac{\rho_2}{\tau_2} \,.\, \sqrt{G},$$

$$\frac{\partial R_1}{\partial v} \cdot \frac{\rho_1}{\tau_1} \,.\, \sqrt{E} - \sqrt{\frac{G}{E}} \cdot \frac{h_1^2}{R_1^2}\left(\frac{\partial R_1}{\partial u} + \sqrt{E}\right)^2 = m\left[\frac{\partial R_2}{\partial u} \cdot \frac{\rho_2}{\tau_2} \sqrt{G} - \sqrt{\frac{E}{G}} \,\cdot\right.$$
$$\left. \cdot\, \frac{h_2^2}{R_2^2}\left(\frac{\partial R_2}{\partial v} + \sqrt{G}\right)^2\right].$$

The first two equations give

$$\frac{\rho_1 \sqrt{E} \,.\, R_1^2}{\tau_1 \,.\, h_1^2 \dfrac{\partial R_1}{\partial v}} = \frac{h_2^2 \cdot \dfrac{\partial R_2}{\partial u} \cdot \tau_2}{\rho_2 \,.\, \sqrt{G} \,.\, R_2^2},$$

or

$$\frac{\tau_1 \tau_2}{\sqrt{EG}} \cdot \frac{\partial R_1}{\partial v} \cdot \frac{\partial R_2}{\partial u} = \frac{\rho_1 \rho_2 R_1^2 R_2^2}{h_1^2 h_2^2}$$

and

$$h_1' h_2' h_1'' h_2'' = \frac{R_1^4 \,.\, R_2^4 \rho_1^2 \rho_2^2}{h_1^4 h_2^4},$$

which is identical with the necessary and sufficient condition that (S_1) and (S_2) have their asymptotic lines corresponding. In case of corresponding lines of curvature this is a necessary condition, but it is not sufficient, as the third equation written above has to be satisfied. We can therefore state: if the lines of curvature on (S_1) and (S_2) correspond, the asymptotic lines correspond also, but not conversely.

§8.

For the spherical representation of an element of a curve on the focal surfaces (S_1) and (S_3) we easily obtain from the general formula:

$$d\sigma_\kappa^2 = (p^{(\kappa)}\,du + p_1^{(\kappa)}\,dv)^2 + (q^{(\kappa)}\,du + q_1^{(\kappa)}\,dv)^2.$$

For (S_1),

$$d\sigma_1^2 = \left(d\varpi_1 + \frac{\sqrt{G}\,dv}{\rho_2}\right)^2 + \frac{h_1^2\,\rho_2^2}{R_1^2\,\rho_1^2}\left(\frac{\sqrt{G}\,dv}{\rho_2}\right)^2.$$

For (S_2),

$$d\sigma_2^2 = \left(d\varpi_2 + \frac{\sqrt{E}\,.\,du}{\rho_1}\right)^2 + \frac{h_2^2\,\rho_1^2}{R_2^2\,\rho_2^2}\,.\left(\frac{\sqrt{E}\,du}{\rho_1}\right)^2,$$

or

$$d\sigma_1^2 = \left(d\varpi_1 + \frac{ds_2}{\rho_2}\right)^2 + \frac{h_1^2}{R_1^2\,\rho_1^2}\,.\,ds_2^2,$$

$$d\sigma_2^2 = \left(d\varpi_2 + \frac{ds_1}{\rho_1}\right)^2 + \frac{h_2^2}{R_2^2\,\rho_2^2}\,.\,ds_1^2.$$

These two formulæ evidently do not depend on the choice of the coordinate lines.

§9.

Let us now consider the congruence of straight lines tangent to any given family of lines on the surface (Σ).

Suppose that the surface is given by $x = f(u,\,v)$, $y = \phi(u,\,v)$, $z = \psi(u,\,v)$. The lines on the surface are given by their differential equation, from which we can obtain the values of $\frac{dv}{du} = \lambda_\kappa$.

Then the coordinates of the focal surface are given by

$$x_1 = x + \alpha_1 R_1;\quad y_1 = y + \beta_1 R_1;\quad z_1 = z + \gamma_1 R_1,$$

where α_1, β_1, γ_1 are the direction cosines of the tangents to the given curve and represented by

$$\alpha_1 = \frac{\dfrac{\partial x}{\partial u} + \lambda_\kappa\,.\,\dfrac{\partial x}{\partial v}}{\sqrt{E + 2F\lambda_\kappa + G\lambda_\kappa^2}} \text{ etc.}$$

R_1 is the focal distance and given according to Kummer, by

$$R_1 = -\frac{gE' - (f+f')F' + eG'}{E'G' - F'^2},$$

where

$$e = \Sigma\frac{\partial x}{\partial u}\cdot\frac{\partial \alpha_1}{\partial u}, \quad f = \Sigma\frac{\partial x}{\partial v}\cdot\frac{\partial x_1}{\partial u}, \quad f' = \Sigma\frac{\partial x}{\partial u}\cdot\frac{\partial x_1}{\partial v}, \quad g = \Sigma\frac{\partial x}{\partial v}\cdot\frac{\partial \alpha_1}{\partial v},$$

$$E' = \Sigma\left(\frac{\partial x_1}{\partial u}\right)^2, \quad F' = \Sigma\frac{\partial x_1}{\partial u}\cdot\frac{\partial x_1}{\partial v}, \quad G' = \Sigma\left(\frac{\partial x_1}{\partial v}\right)^2.$$

The fundamental coefficients of the focal surface $\overline{E}$, $\overline{F}$, $\overline{G}$ we obtain according to the formulæ:

$$\overline{E} = \Sigma\left(\frac{\partial x_1}{\partial u}\right)^2 = \Sigma\left(\frac{\partial x}{\partial u}\right)^2 + R_1^2\Sigma\left(\frac{\partial x_1}{\partial u}\right)^2 + \left(\frac{\partial R_1}{\partial u}\right)^2 + 2R_1\Sigma\frac{\partial x}{\partial u}\cdot\frac{\partial x_1}{\partial u}$$
$$+ 2\frac{\partial R_1}{\partial u}\cdot\Sigma\frac{\partial x}{\partial u}\cdot\alpha_1, \text{ etc.,}$$

or

$$\overline{E} = E + R_1^2E' + \left(\frac{\partial R_1}{\partial u}\right)^2 + 2R_1e + 2\frac{\partial R_1}{\partial u}\Sigma\alpha_1\cdot\frac{\partial x}{\partial u},$$

$$\overline{F} = F + R_1^2F' + \frac{\partial R_1}{\partial u}\cdot\frac{\partial R_1}{\partial v} + R_1(f+f') + \frac{\partial R_1}{\partial v}\Sigma\alpha_1\cdot\frac{\partial x}{\partial u} + \frac{\partial R_1}{\partial u}\Sigma\alpha_1\cdot\frac{\partial x}{\partial v},$$

$$\overline{G} = G + R_1^2G' + \left(\frac{\partial R_1}{\partial v}\right)^2 + 2R_1g + 2\cdot\frac{\partial R_1}{\partial v}\Sigma\alpha_1\cdot\frac{\partial x}{\partial v}.$$

The linear element ds_1 is therefore

$$ds_1^2 = ds^2 + R_1^2d\sigma_1^2 + dR_1^2 + 2R_1\left[edu^2 + (f+f')\,du\,dv + g'\,dv^2\right]$$
$$+ 2dR_1\{\cos V\cdot\sqrt{E}\cdot du + \cos V_1\cdot\sqrt{G}\cdot dv\},$$

where $\cos V = \frac{1}{\sqrt{E}}\cdot\Sigma\alpha_1\cdot\frac{\partial x}{\partial u}$, $\cos V_1 = \frac{1}{\sqrt{G}}\cdot\Sigma\alpha_1\cdot\frac{\partial x}{\partial v}$.

V and V_1 are the angles between the direction R_1 and the lines $v =$ const. and $u =$ const. respectively.

It remains to express

$$d\sigma_1^2 = E'\,du^2 + 2F'\,du\,dv + G'\,dv^2.$$

For this purpose imagine a sphere of radius 1 and through the center of it draw the lines parallel to the direction R_1; they will describe a certain line on the sphere, the coordinates of which are α_1, β_1, γ_1 and $d\sigma_1^2$ is the element of arc of what is called the spherical indicatrix of the given line. Its value can be written up from the fact that

$$\frac{d\sigma_1}{ds_\kappa} = \frac{1}{h_\kappa},$$

where h_κ is the radius of the first curvature of the given curve and ds_κ is the element of arc of this curve given by

$$ds_\kappa^2 = E + 2F\lambda_\kappa + G\lambda_\kappa^2.$$

Hence

$$d\sigma_1^2 = \frac{ds_\kappa^2}{h_\kappa^2}.$$

The expression

$$edu^2 + (f + f')\,du\,dv + gdv^2$$

is identically zero. For

$$e = \frac{1}{ds_\kappa}\Big[mdu + m'dv + E\frac{\partial}{\partial u}(du) + F\frac{\partial}{\partial u}(dv) - \frac{1}{ds_\kappa}(E\,du + F\,dv)\frac{\partial}{\partial u}(ds_\kappa)\Big],$$

$$f + f' = \frac{1}{ds_\kappa}\Big[m'du + m''dv + E\,.\frac{\partial}{\partial v}(du) + F\,.\frac{\partial}{\partial v}(dv) - \frac{1}{ds_\kappa}(Edu + Fdv)\,.\frac{\partial}{\partial v}(ds_\kappa) + ndu + n'dv + F\,.\frac{\partial}{\partial u}(du) + G\frac{\partial}{\partial v}(dv) - \frac{1}{ds_\kappa}(Fdu + Gdv)\frac{\partial}{\partial u}(ds_\kappa)\Big],$$

$$g = \frac{1}{ds_\kappa}\Big[n'du + n''dv + F\frac{\partial}{\partial v}(du) + G\,.\frac{\partial}{\partial v}(dv) - \frac{1}{ds_\kappa}(Fdu + Gdv)\frac{\partial}{\partial v}(ds_\kappa)\Big],$$

and hence

$$edu^2 + (f + f')\,du\,dv + gdv^2 = \frac{1}{ds_\kappa}\big[du\,(mdu^2 + 2m'\,du\,dv + m''dv^2 + Ed^2u + Fd^2v) + dv\,(ndu^2 + 2n'du\,dv + n''dv^2 + Fd^2u + Gd^2v\big] - d^2s_\kappa \equiv 0,$$

where

$$m = \Sigma\frac{\partial x}{\partial u}\cdot\frac{\partial^2 x}{\partial u^2},\quad m' = \Sigma\frac{\partial x}{\partial u}\cdot\frac{\partial^2 x}{\partial u\partial v},\quad m'' = \Sigma\frac{\partial x}{\partial u}\cdot\frac{\partial^2 x}{\partial v^2},$$

$$n = \Sigma\frac{\partial x}{\partial v}\cdot\frac{\partial^2 x}{\partial u^2},\quad n' = \Sigma\frac{\partial x}{\partial v}\cdot\frac{\partial^2 x}{\partial u\partial v},\quad n'' = \Sigma\frac{\partial^2 x}{\partial v^2}\cdot\frac{\partial x}{\partial v}.$$

Again,
$$\cos V = \frac{E du + F dv}{\sqrt{E} . ds_\kappa},$$
$$\cos V_1 = \frac{F du + G dv}{\sqrt{G} . ds_\kappa}.$$

Hence the linear element of the focal surface becomes
$$ds_1'^2 = ds_\kappa^2 + R_1^2 . \frac{ds_\kappa^2}{h_\kappa^2} + dR_1^2 + 2 dR_1 . ds_\kappa .$$

From this expression we see that for any coordinates u and v
$$\overline{E} = \left(1 + \frac{R_1^2}{h_\kappa^2}\right)\left(\frac{\partial s_\kappa}{\partial u}\right)^2 + \left(\frac{\partial R_1}{\partial u}\right)^2 + 2 . \frac{\partial R_1}{\partial u} . \frac{\partial s_\kappa}{\partial u},$$
$$\overline{F} = \left(1 + \frac{R_1^2}{h_\kappa^2}\right)\frac{\partial s_\kappa}{\partial u} . \frac{\partial s_\kappa}{\partial v} + \frac{\partial R_1}{\partial u} . \frac{\partial R_1}{\partial v} + \frac{\partial R_1}{\partial v} . \frac{\partial s_\kappa}{\partial u} + \frac{\partial R_1}{\partial u} . \frac{\partial s_\kappa}{\partial v},$$
$$\overline{G} = \left(1 + \frac{R_1^2}{h_\kappa^2}\right)\left(\frac{\partial s_\kappa}{\partial v}\right)^2 + \left(\frac{\partial R_1}{\partial v}\right) + 2 . \frac{\partial R_1}{\partial v} . \frac{\partial s_\kappa}{\partial v} .$$

If we write the above expression for ds_1^2 as follows:
$$ds_1'^2 = (ds_\kappa + dR_1)^2 + \frac{R_1^2}{h_\kappa^2} . ds_\kappa^2 = dq^2 + \frac{R_1^2}{h_\kappa^2} ds_\kappa^2,$$
where
$$q = s_\kappa + R_1,$$
we see that
$$s_\kappa = \text{const.}$$
represents a system of geodesics on the focal surface, and
$$s_\kappa + R_1 = \text{const.}$$
are their orthogonal trajectories.

Suppose the surface (Σ) is referred to a system of geodesic lines and their orthogonal trajectories; then $E = 1$, $F = 0$; consider the congruence of lines tangent to the geodesics $v = \text{const.}$; $s_\kappa = u = \text{const.}$ will be geodesics on the focal surface, and since
$$\frac{1}{R_1} = - \frac{\partial \lg \sqrt{G}}{\partial u},$$
their orthogonal trajectories are
$$u - 1 : \frac{\partial \lg \sqrt{G}}{\partial u} = \text{const.}$$
We perceive at once that if
$$G = f(u),$$

i. e. if our surface (Σ) is applicable to a surface of revolution, $R_1 + u = f_1(u)$, and therefore the focal surface cannot be referred to the system of coordinates

$$R_1 + s_\kappa \text{ and } s_\kappa,$$

for the lines

$$R_1 + s_\kappa = \text{const. and } s_\kappa = \text{const.}$$

are identical.

Excluding this case from consideration, we see that since

$$u - \frac{1}{\frac{\partial \lg \sqrt{G}}{\partial u}} = \text{const.} = a,$$

by integrating we obtain

$$\sqrt{G} \, . \, V = u - a,$$

and without any loss of generality we can make $V = 1$. Hence

$$\sqrt{G} = u - a$$

are orthogonal trajectories of the geodesics $u =$ const.

In the case we have excluded from consideration we can introduce a new parameter, namely,

$$\frac{ds_\kappa}{h_\kappa} = dw,$$

where dw is the curvature of the geodesic $v =$ const. Then

$$ds_1^2 = d(s_\kappa + R_1)^2 + R_1^2 \, . \, dw^2,$$

and we see that the focal surface is also applicable to a surface of revolution. This is the well-known Weingarten's proposition.

§10.

We can state without proof, which is altogether similar to that given on page 110, etc., the following theorem:

If the edges of regression of one system of the developable surfaces of the congruence of lines are such that their radii of curvature are functions of the corresponding focal distances, then the second focal surface is applicable to a surface of revolution. We can find all the isometric systems on it by simple quadratures, also can determine all the geodesics which can be drawn on it by simple quadratures. Furthermore, if for different congruences the radii of curvature above mentioned are the same functions of the corresponding focal distances, the second focal surfaces for these congruences are all applicable to one another. If the radii of curvature are functions of the arcs of the edges of regression, the second focal surface is a developable surface.

We give here also without proof the formulæ for direction cosines of the normals to the second focal surface. The proof is based upon the theorem of Koenigs given on page 103. Calling X_2, Y_2, Z_2 these direction cosines, we have

$$X_2 = -\frac{h_\kappa X}{R_\kappa} + \frac{h_\kappa}{T\rho_\kappa}\left[\frac{\partial x}{\partial u}\cdot\frac{F+G\lambda}{\sqrt{E+2F\lambda+G\lambda^2}} - \frac{\partial x}{\partial v}\cdot\frac{E+F\lambda}{\sqrt{E+2F\lambda+G\lambda^2}}\right],$$

$$Y_2 = -\frac{h_\kappa Y}{R_\kappa} + \frac{h_\kappa}{T\rho_\kappa}\left[\frac{\partial y}{\partial u}\cdot\frac{F+G\lambda}{\sqrt{E+2F\lambda+G\lambda^2}} - \frac{\partial y}{\partial v}\cdot\frac{E+F\lambda}{\sqrt{E+2F\lambda+G\lambda^2}}\right],$$

$$Z_2 = -\frac{h_\kappa}{R_\kappa}.Z + \frac{h_\kappa}{T\rho_\kappa}\left[\frac{\partial z}{\partial u}\cdot\frac{F+G\lambda}{\sqrt{E+2F\lambda+G\lambda^2}} - \frac{\partial z}{\partial v}\cdot\frac{E+F\lambda}{\sqrt{E+2F\lambda+G\lambda^2}}\right].$$

Here R_κ is the radius of geodesic curvature of the edge of regression, ρ_κ— radius of principal curvature of the first focal surface, $\lambda = \frac{dv}{du}$ is taken from the differential equation of the lines the tangents to which we consider in the congruence,

$$T^2 = EG - F^2.$$

§11.

Here we want to state the following theorem which escaped us at the time of writing the above matter. Namely, on the page 114 we proved the following theorem: if there exists a relation

$$h_1 = f(R_1),$$

then the lines connecting the centers of geodesic curvature of the u-line with the corresponding centers of principal curvature of the v-line form a congruence of rays normal to a surface. These rays all lie in the tangent planes to the focal surface (S_1), and therefore are all tangent to this surface, i. e. this surface is the surface of the centers of the surface to which all the rays are normal. Now since this surface of the centers is applicable to a surface of revolution, it must be a Weingarten's surface, i. e. its radii of principal curvatures are functions of one another.

The linear element we reduced in this case to the form

$$ds_1'^2 = dv_2^2 + \frac{1}{V_2} \cdot du_1^2.$$

We can take v_2 for the radius ρ_{σ_1} of the surface to which the rays are normal. But

$$\begin{aligned} v_2 &= \int \frac{dv_1}{\sqrt{V_1}} = \int \frac{(1 - \phi^2(R_1'))^{\frac{1}{2}}\, dR_1'}{\phi(R_1')} \\ &= F(R_1). \end{aligned}$$

Hence
$$R_1 = \text{const.}$$
is the line of equal curvature of the above surface, and
$$u_1 = \text{const.}$$
its lines of curvature of one system.

Again,
$$u_1 = s_1 + F_1(R_1).$$

If s_1 be a function of R_1, we cannot take $s_1 + R_1$ and R_1 as the parameters of the coordinate lines on the surface (S_1).

But then as before
$$ds_1'^2 = d(s_1 + R_1)^2 + R_1^2\, dw^2,$$
where
$$dw = \frac{ds_1}{h_1}.$$

In this case the focal surface is applicable to a surface of revolution.